Synthesis of Transformerless Active N-Port Networks

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The following theorem is proved:

Theorem: An arbitrary symmetric $N \times N$ matrix of real rational functions in the complex-frequency variable (a) can be realized as the immittance matrix of an N-port network containing only resistors, capacitors, and N negative-RC impedances, and (b) cannot, in general, be realized as the immittance matrix of an N-port network containing resistors, capacitors, inductors, ideal transformers, and M negative-RC impedances if $M < N$.

The necessary and sufficient conditions for the immittance-matrix realization of transformerless networks of capacitors, self-inductors, resistors, and negative resistors follow as a special case of the theorem. In addition, an earlier result is extended by presenting a procedure for the realization of an arbitrary $N \times N$ short-circuit admittance matrix as an unbalanced transformerless active RC network requiring no more than N controlled sources. The passive RC structure has the interesting property that it can always be realized as a $(3N + 1)$-terminal network of two-terminal impedances with common reference node and no internal nodes. The active subnetwork can always be realized with N negative-impedance converters.

I. INTRODUCTION

The development of the transistor has provided the network synthesist with an efficient low-cost active element and has stimulated considerable interest in the theory of active RC networks during the last decade.

Several techniques have been proposed for the transformerless active RC realization of transfer and driving-point functions.[1-18] It has, in fact, been established that any real rational fraction (in the complex frequency variable) can be realized as the transfer or driving-point function of a transformerless active RC network containing one active element. In particular, Linvill's technique[3] has been the basis for much of the later work.

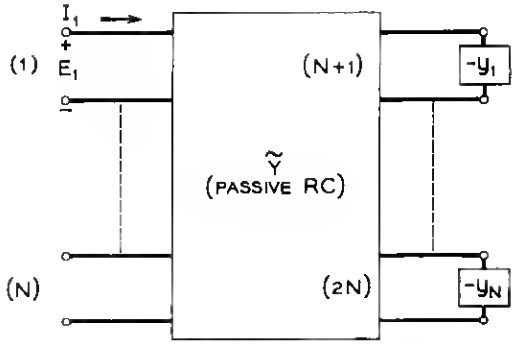

Fig. 1 — Realization of an arbitrary $N \times N$ symmetric immittance matrix.

It has recently been shown[19] that an arbitrary $N \times N$ matrix of real rational functions can be realized as the short-circuit admittance matrix of a transformerless N-port active RC network containing N controlled sources, and that in general all N controlled sources are required. These results have suggested the possibility of establishing the theorem stated in the abstract to this paper. The proof, presented in the next section, is based on a technique developed in an earlier paper for factoring a class of matrix-coefficient polynomials in a scalar variable. For the special case $N = 1$, our result reduces to that of Sipress.[18] *

We also present in Section II a procedure for the realization of an arbitrary $N \times N$ short-circuit admittance matrix as an unbalanced active RC network requiring no more than N controlled sources. The required passive RC network has the interesting property that it can always be realized as a $(3N + 1)$-terminal network of two-terminal impedances with common reference node and no internal nodes. This result not only displaces the balanced network assumption implicit in the proof given in Ref. 19, but is of considerable interest in its own right.

II. REALIZATION OF A SYMMETRIC IMMITTANCE MATRIX AS AN ACTIVE *RC* NETWORK CONTAINING NEGATIVE-*RC* IMPEDANCES

Consider a $2N$-port network of resistors and capacitors characterized by the short-circuit admittance matrix $\tilde{\mathbf{Y}}$ and suppose that a negative-RC admittance $-y_k$ is connected to port $N + k$ $(k = 1,2,\cdots,N)$, as shown in Fig. 1. It is convenient to partition $\tilde{\mathbf{Y}}$ as follows:

* This case was first considered in detail by Kinariwala,[13] who showed that a broad class of driving-point functions could be realized.

$$\tilde{\mathbf{Y}} = \begin{array}{cc} \begin{array}{cc} N & N \end{array} & \\ \begin{bmatrix} \mathbf{Y}_{11} & \mathbf{Y}_{12} \\ \mathbf{Y}_{21} & \mathbf{Y}_{22} \end{bmatrix} & \begin{array}{c} N \\ N \end{array} \end{array}. \tag{1}$$

The short-circuit admittance matrix $\mathbf{Y}$ relating the voltages and currents at ports $k(k = 1,2,\cdots,N)$ can readily be shown to be

$$\mathbf{Y} = \mathbf{Y}_{11} - \mathbf{Y}_{12}[\mathbf{Y}_{22} - \text{diag}\ (y_1,y_2,\cdots,y_N)]^{-1}\mathbf{Y}_{12}{}^t, \tag{2}$$

where the superscript t indicates matrix transposition.

We assume that $\mathbf{Y} = (1/D)[N_{ij}]$ is an arbitrary prescribed symmetric $N \times N$ matrix of real rational functions, where $[N_{ij}]$ is a matrix of polynomials and D is a common denominator polynomial. The synthesis technique requires that the three submatrices in (2) be determined so that $\tilde{\mathbf{Y}}$ is realizable as a transformerless RC network and that the elements in diag $(y_1,y_2,\cdots,y_N)$ be RC driving-point admittances.

The matrix $\tilde{\mathbf{Y}}$ can be expressed as

$$\tilde{\mathbf{Y}} = s\mathbf{K}_\infty + \sum_{m=0}^{M} \mathbf{K}_m \frac{s}{s + \gamma_m}, \tag{3}$$

where $\mathbf{K}_\infty$ and $\mathbf{K}_m$ are real symmetric coefficient matrices and the γ_m are real and satisfy

$$0 = \gamma_0 < \gamma_1 < \gamma_2 \cdots < \gamma_M . \tag{4}$$

It is well known that, if the coefficient matrices in (3) are "dominant-diagonal" matrices,* $\tilde{\mathbf{Y}}$ can be realized as a transformerless balanced RC network.[20] Our objective is to determine the submatrices in (1) so that $\tilde{\mathbf{Y}}$ satisfies the dominant-diagonal condition. To simplify the discussion it is assumed that $\tilde{\mathbf{Y}}$ is to be regular at infinity.

2.1 *The Synthesis Technique*

Consider the class of matrices $\mathbf{Y}_{11}$, $\mathbf{Y}_{12}$, $\mathbf{Y}_{22}$, and diag $(y_1,y_2,\cdots,y_N)$ satisfying (2) such that $\mathbf{Y}_{12}$ and $[\mathbf{Y} - \mathbf{Y}_{11}]$ possess inverses. As a first step in obtaining insight into the realization problem we rewrite (2) in the following form:

$$-\mathbf{Y}_{12}{}^t[\mathbf{Y} - \mathbf{Y}_{11}]^{-1}\mathbf{Y}_{12} = \mathbf{Y}_{22} - \text{diag}\ (y_1,y_2,\cdots,y_N). \tag{5}$$

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* A dominant-diagonal matrix M has elements m_{jk} which satisfy

$$m_{jj} \geqq \sum_{k \neq j} | m_{jk} |.$$

It is convenient to employ the following notation:

$$\begin{aligned} \mathbf{Y}_{11} &= \frac{1}{q}[x_{ij}] = \frac{1}{q}\mathbf{X}_{11}, \\ \mathbf{P} &= [qN_{ij} - Dx_{ij}], \\ \mathbf{Y}_{12} &= \frac{1}{q}\mathbf{X}_{12}, \end{aligned} \tag{6}$$

where $\mathbf{X}_{11}$, $\mathbf{P}$, and $\mathbf{X}_{12}$ are $N \times N$ matrices of polynomials and q is a common denominator polynomial.

From (5) and (6),

$$-\frac{D}{q}\mathbf{X}_{12}{}^t\mathbf{P}^{-1}\mathbf{X}_{12} = \mathbf{Y}_{22} - \mathrm{diag}\,(y_1,y_2,\cdots,y_N). \tag{7}$$

The left-hand side of (7) can be written before cancellation of common factors as a matrix of real rational functions with common denominator polynomial q det $\mathbf{P}$. Since the poles of the right-hand side of (7) are required to be distinct and on the negative-real axis, $\mathbf{X}_{12}$ must be chosen so that the least common denominator polynomial of the matrix of rational functions has only zeros that are distinct and on the negative-real axis. To satisfy this condition, we employ a matric polynomial factorization technique developed in an earlier paper.[19] Specifically, it is shown in Appendix A that, given $\mathbf{Y}$, a realizable submatrix $\mathbf{Y}_{11} = (1/q)\,[x_{ij}]$ can be chosen so that:

(a) deg x_{ii} = deg $q = NL_0(i = 1,2,\cdots,N)$, where* $L_0 = \max\,[\max \deg N_{ij}, \deg D]$;

(b) the off-diagonal numerator polynomials $x_{ij}(i \neq j)$ are any set of real polynomials consistent with $x_{ij} = x_{ji}$ and deg $x_{ij} \leqq$ deg q;

(c) $\mathbf{Y}_{11}$ has only coefficient matrices that satisfy the dominant-diagonal condition with the inequality sign;

(d) the matric polynomial $\mathbf{P}$ [defined in (6)], of degree* deg $q + L_0$ can be written as the product $\mathbf{P}_1\mathbf{P}_2$ of two matric polynomials $\mathbf{P}_1$ and $\mathbf{P}_2$ (with $N \times N$ matrix coefficients) of degrees respectively deg q and L_0;

(e) det $\mathbf{P}$ does not vanish identically; and

(f) the matric polynomial $\mathbf{P}_2$ has the property that det $\mathbf{P}_2$, a polynomial of degree NL_0, has only distinct negative-real zeros that are different from those of q.

In that which follows, we shall assume that conditions (a) through (f) are satisfied.

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* The degree requirement is merely a sufficient condition.

In accordance with (d) and (f), note that the left-hand side of (7) can have only distinct negative-real poles if $\mathbf{X}_{12}$ is chosen to be $(1/\alpha)\mathbf{P}_1$, where α is any nonzero real constant, for then (7) reduces to*

$$\frac{-D}{\alpha^2 q \det \mathbf{P}_2} \mathbf{P}_1{}^t \operatorname{adj} \mathbf{P}_2 = \mathbf{Y}_{22} - \operatorname{diag}(y_1,y_2,\cdots,y_N). \tag{8}$$

In addition, with this choice of $\mathbf{X}_{12}$, $\mathbf{Y}_{12}$ is regular at infinity [see (6) and (d)]. Therefore, by choosing the magnitude of α sufficiently large it is always possible [see (e)] to satisfy the dominant-diagonal condition for the first N rows of $\tilde{\mathbf{Y}}$. Hence let

$$\mathbf{Y}_{12} = \frac{1}{\alpha q}\mathbf{P}_1. \tag{9}$$

It remains to identify $\mathbf{Y}_{22}$ and the y_i such that the dominant-diagonal condition can be satisfied in the last N rows of $\tilde{\mathbf{Y}}$.

The left-hand side of (8) also is regular at infinity since the required condition:

$$\deg D + \deg \mathbf{P}_1 + \deg \operatorname{adj} \mathbf{P}_2 \leqq \deg q + NL_0 \tag{10}$$

reduces to

$$\deg D \leqq \max\,[\max \deg N_{ij}, \deg D]. \tag{11}$$

From (f),

$$q \det \mathbf{P}_2 = \lambda \prod_{m=1}^{M} (s + \gamma_m), \tag{12}$$

where λ is a nonzero real constant, $M = \deg q + NL_0$, and

$$0 < \gamma_1 < \gamma_2 \cdots < \gamma_M .$$

In view of (10) and (12), (8) can be rewritten as

$$\mathbf{Y}_{22} - \operatorname{diag}(y_1,y_2,\cdots,y_N) = \sum_{m=0}^{M} \mathbf{A}_m \frac{s}{s+\gamma_m}, \tag{13}$$

where

$$0 = \gamma_0 < \gamma_1 < \gamma_2 \cdots < \gamma_M$$

and the $\mathbf{A}_m$ are real symmetric coefficient matrices. It is clear from (13) that each off-diagonal term in $\mathbf{Y}_{22}$ is equal to the corresponding sum on

* In (8), adj $\mathbf{P}_2$ refers to the adjoint of $\mathbf{P}_2$ which is defined by $\mathbf{P}_2 \operatorname{adj} \mathbf{P}_2 = \mathbf{U} \det \mathbf{P}_2$, where $\mathbf{U}$ is the identity matrix.

the right-hand side and that

$$\operatorname{diag}(\tilde{y}_{N+1,N+1},\tilde{y}_{N+2,N+2},\cdots,\tilde{y}_{2N,2N}) - \operatorname{diag}(y_1,y_2,\cdots,y_N)$$
$$= \sum_{m=0}^{M} \frac{s}{s+\gamma_m} \operatorname{diag}(a_{11m},a_{22m},\cdots,a_{NNm}). \tag{14}$$

Let

$$\operatorname{diag}(a_{11m},a_{22m},\cdots,a_{NNm})$$
$$= \operatorname{diag}(b_{11m},b_{22m},\cdots,b_{NNm}) - \operatorname{diag}(c_{11m},c_{22m},\cdots,c_{NNm}),$$

where

$$b_{iim},c_{iim} \geqq 0(i = 1,2,\cdots,N).$$

The $\tilde{y}_{N+i,N+i}$ and y_i can be identified as follows:

$$\operatorname{diag}(\tilde{y}_{N+1,N+1},\tilde{y}_{N+2,N+2},\cdots,\tilde{y}_{2N,2N})$$
$$= \sum_{m=0}^{M} \frac{s}{s+\gamma_m} \operatorname{diag}(b_{11m}+d_{11m},\, b_{22m}+d_{22m},\, \cdots,\, b_{NNm}+d_{NNm}), \tag{15}$$

$$\operatorname{diag}(y_1,y_2,\cdots,y_N)$$
$$= \sum_{m=0}^{M} \frac{s}{s+\gamma_m} \operatorname{diag}(c_{11m}+d_{11m},\, c_{22m}+d_{22m},\, \cdots,\, c_{NNm}+d_{NNm}), \tag{16}$$

where the matrices $\operatorname{diag}(d_{11m},d_{22m},\cdots,d_{NNm})$ are chosen to satisfy the dominant-diagonal condition in the last N rows of $\tilde{\mathbf{Y}}$. Hence the matrix $\tilde{\mathbf{Y}}$ is realizable as a transformerless balanced $2N$-port RC network for all symmetric $N \times N$ matrices $\mathbf{Y}$ of real rational functions.

The realization of an arbitrary symmetric open-circuit impedance matrix $\mathbf{Z}$ can be treated as follows. The elements of a matrix $\mathbf{R} = \operatorname{diag}(r_1,r_2,\cdots,r_N)$ can be chosen nonnegative and sufficiently large so that $\mathbf{Y}' = [\mathbf{Z} - \mathbf{R}]^{-1}$ exists. Therefore, $\mathbf{Z}$ can be realized by inserting a (nonnegative) resistor r_k in series with each port $k(k = 1,2,\cdots,N)$ of a network characterized by $\mathbf{Y}'$.*

The proof relating to the necessity of N negative-RC admittances follows directly from a more general result developed previously.[19] †

* Similarly, the theorem proved in Ref. 19 remains valid if the words "short-circuit admittance" are replaced with "open-circuit impedance."

† In connection with the analysis in Ref. 19, it is worthwhile to point out that any controlled voltage (current) source can be replaced with an arbitrarily chosen finite impedance (admittance) in series (parallel) with a new controlled voltage (current) source whose output differs from that of the original source by a term which nullifies the effect of the impedance (admittance). With this understanding, it is not necessary to consider further the degenerate cases which can arise if zero and/or infinite impedance paths appear when the controlled sources are set equal to zero.

The techniques presented in this section bear heavily on the problem of realizing unbalanced transformerless N-port active RC networks. These considerations are treated in detail in the following section.

III. UNBALANCED ACTIVE RC REALIZATION OF AN ARBITRARY SHORT-CIRCUIT ADMITTANCE MATRIX

We consider a $(3N + 1)$-terminal RC network to which is connected at terminals $N + k$ $(k = 1,2,\cdots,2N)$ and the common reference node a $(2N + 1)$-terminal active network as shown in Fig. 2. Denote by $\mathbf{E}_a$, $\mathbf{E}_b$, $\mathbf{E}_c$, $\mathbf{I}_a$, $\mathbf{I}_b$, and $\mathbf{I}_c$ the following column matrices of voltages and currents:

$$\mathbf{E}_a = \begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_N \end{bmatrix}, \quad \mathbf{E}_b = \begin{bmatrix} E_{N+1} \\ E_{N+2} \\ \vdots \\ E_{2N} \end{bmatrix}, \quad \mathbf{E}_c = \begin{bmatrix} E_{2N+1} \\ E_{2N+2} \\ \vdots \\ E_{3N} \end{bmatrix},$$
$$\mathbf{I}_a = \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_N \end{bmatrix}, \quad \mathbf{I}_b = \begin{bmatrix} I_{N+1} \\ I_{N+2} \\ \vdots \\ I_{2N} \end{bmatrix}, \quad \mathbf{I}_c = \begin{bmatrix} I_{2N+1} \\ I_{2N+2} \\ \vdots \\ I_{3N} \end{bmatrix}. \tag{17}$$

It is convenient to partition $\hat{\mathbf{Y}}$, the short-circuit admittance matrix of

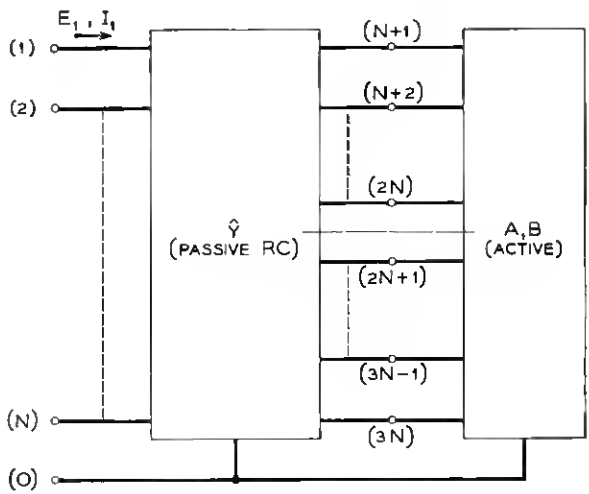

Fig. 2 — Unbalanced realization of an arbitrary $N \times N$ short-circuit admittance matrix.

the $(3N + 1)$-terminal network, as follows:

$$\hat{\mathbf{Y}} = \begin{array}{cc} \begin{array}{ccc} N & N & N \end{array} & \\ \begin{bmatrix} \mathbf{Y}_{11} & \mathbf{Y}_{12} & \mathbf{Y}_{13} \\ \mathbf{Y}_{21} & \mathbf{Y}_{22} & \mathbf{Y}_{23} \\ \mathbf{Y}_{31} & \mathbf{Y}_{32} & \mathbf{Y}_{33} \end{bmatrix} & \begin{array}{c} N \\ N \\ N \end{array} \end{array}. \tag{18}$$

The active network is assumed to impose the constraints*

$$\begin{aligned} \mathbf{I}_b &= -\mathbf{A}\mathbf{I}_c , \\ \mathbf{E}_c &= -\mathbf{B}\mathbf{E}_b , \end{aligned} \tag{19}$$

where $\mathbf{A}$ and $\mathbf{B}$ are $N \times N$ coefficient matrices. It is not difficult to derive the following expression for the short-circuit admittance matrix $\mathbf{Y}$ relating $\mathbf{E}_a$ and $\mathbf{I}_a$, the voltages and currents at the N accessible ports in Fig. 2:

$$\begin{aligned} \mathbf{Y} = \mathbf{Y}_{11} &+ (\mathbf{Y}_{12} - \mathbf{Y}_{13}\mathbf{B}) \\ &\cdot [\mathbf{A}\mathbf{Y}_{33}\mathbf{B} - \mathbf{Y}_{22} - \mathbf{A}\mathbf{Y}_{32} + \mathbf{Y}_{32}{}^t\mathbf{B}]^{-1}(\mathbf{Y}_{12}{}^t + \mathbf{A}\mathbf{Y}_{13}{}^t). \end{aligned} \tag{20}$$

We shall simplify the discussion by assuming that the matrices $\mathbf{A}$ and $\mathbf{B}$ are given by

$$\mathbf{A} = a\mathbf{U}, \qquad \mathbf{B} = b\mathbf{U}, \tag{21}$$

where $\mathbf{U}$ is the identity matrix of order N and a and b are real constants such that $ab > 0$. The synthesis technique does not further restrict the choice of a and b so that the $(2N + 1)$-terminal active network can always be realized with N voltage-inversion or N current-inversion negative-impedance converters by choosing respectively $a = 1$, $b > 0$ or $b = -1$, $a < 0$. Note therefore that the realization can always be accomplished with N controlled sources.

We shall consider explicitly the case in which $a, b > 0$ and indicate the modifications necessary to treat the remaining case.

Our objective is to prove for all prescribed matrices $\mathbf{Y}$ that $\hat{\mathbf{Y}}$ can be realized as a $(3N + 1)$-terminal network of two-terminal impedances with common reference node and no internal nodes. It is well known[20] that the necessary and sufficient conditions for achieving this type of realization are that the coefficient matrices in

$$\hat{\mathbf{Y}} = s\mathbf{K}_\infty + \sum_{m=0}^{M} \mathbf{K}_m \frac{s}{s + \gamma_m} \tag{22}$$

* The matrices $\mathbf{A}$ and $\mathbf{B}$ should not be confused with the diagonal matrices $\mathbf{A}_m$ and $\mathbf{B}_m$ introduced in Section 2.1.

be real symmetric dominant-diagonal matrices with nonpositive off-diagonal terms, and

$$0 = \gamma_0 < \gamma_1 < \gamma_2 \cdots < \gamma_M , \qquad \gamma_m \text{ real.} \tag{23}$$

It is clear that all off-diagonal terms in $\hat{\mathbf{Y}}$ are required to be negative-RC driving-point admittance functions. For simplicity we assume that $\hat{\mathbf{Y}}$ is not to have a pole at infinity ($\mathbf{K}_\infty = 0$).

3.1 *The Realization Technique*

Our notation is identical to that used in the preceding Section 2.1:

$$\begin{aligned} \mathbf{Y}_{11} &= \frac{1}{q}[x_{ij}] = \frac{1}{q}\mathbf{X}_{11}, \qquad & \mathbf{P} &= [qN_{ij} - Dx_{ij}], \\ \mathbf{Y}_{12} &= \frac{1}{q}\mathbf{X}_{12}, & \mathbf{Y}_{13} &= \frac{1}{q}\mathbf{X}_{13}. \end{aligned} \tag{24}$$

By paralleling the development in Section 2.1* and using (20), (21), and (24), we obtain†

$$\frac{D}{q}(\mathbf{X}_{12}^{t} + a\mathbf{X}_{13}^{t})\mathbf{P}^{-1}(\mathbf{X}_{12} - b\mathbf{X}_{13}) = ab\mathbf{Y}_{33} - \mathbf{Y}_{22} - a\mathbf{Y}_{32} + b\mathbf{Y}_{32}^{t}. \tag{25}$$

We again assume that $\mathbf{Y}_{11}$ is chosen so that (a) through (f) (Section 2.1) are satisfied. It is assumed in addition that the off-diagonal terms in $\mathbf{Y}_{11}$ are chosen to be negative-RC driving-point admittance functions [see (b)].

Next let

$$\begin{aligned} \mathbf{X}_{12} - b\mathbf{X}_{13} &= \frac{1}{\beta_1}\mathbf{P}_1 , \\ \mathbf{X}_{12}^{t} + a\mathbf{X}_{13}^{t} &= \frac{1}{\beta_2}\mathbf{P}_3 , \end{aligned} \tag{26}$$

where β_1 and β_2 are nonzero real parameters to be chosen in accordance with the discussion below and $\mathbf{P}_3$ is a nonsingular matrix of N^2 polynomials chosen so that each entry in $(1/q)\mathbf{P}_3$ is a negative-RC driving-point admittance function that is nonzero at the origin and finite at infinity. It is clear that $\deg \mathbf{P}_3 = \deg q$.

We consider the matrices $\mathbf{Y}_{12}$ and $\mathbf{Y}_{13}$. From (24) and (26) we find

* It is assumed that $[\mathbf{Y}_{12} - b\mathbf{Y}_{13}]$, $[\mathbf{Y} - \mathbf{Y}_{11}]$, and $[\mathbf{Y}_{12}^{t} + a\mathbf{Y}_{13}^{t}]$ possess inverses.

† The writer is indebted to J. M. Sipress for suggesting a study of (25) by exploiting the essential similarities between it and (7).

$$\begin{aligned} \mathbf{Y}_{12} &= \frac{1}{q} \frac{b}{\beta_2(a+b)} \left[\mathbf{P}_3{}' + \frac{a\beta_2}{b\beta_1} \mathbf{P}_1 \right], \\ \mathbf{Y}_{13} &= \frac{1}{q} \frac{1}{\beta_2(a+b)} \left[\mathbf{P}_3{}' - \frac{\beta_2}{\beta_1} \mathbf{P}_1 \right]. \end{aligned} \tag{27}$$

Suppose that* $a,b,\beta_2 > 0$. Note that, since $\deg \mathbf{P}_1 = \deg \mathbf{P}_3 = \deg q$, it is possible to choose $|\beta_2/\beta_1|$ sufficiently small such that each element in $\mathbf{Y}_{12}$ and $\mathbf{Y}_{13}$ is a negative-RC driving-point admittance function. It is clear that this ratio can be held invariant while β_2 is chosen sufficiently large to satisfy the dominant-diagonal condition in the first N rows of $\hat{\mathbf{Y}}$.

At this point the synthesis problem reduces to the determination of the submatrices $\mathbf{Y}_{23}$, $\mathbf{Y}_{33}$, and $\mathbf{Y}_{22}$.

3.2 *Determination of* $\mathbf{Y}_{23}$, $\mathbf{Y}_{33}$, *and* $\mathbf{Y}_{22}$

Substituting (26) into (25) gives

$$\frac{1}{\beta_1\beta_2} \frac{D}{q} \mathbf{P}_3\mathbf{P}_2^{-1} = ab\mathbf{Y}_{33} - \mathbf{Y}_{22} - a\mathbf{Y}_{32} + b\mathbf{Y}_{32}{}', \tag{28}$$

where

$$q \det \mathbf{P}_2 = \lambda \prod_{m=1}^{M} (s + \gamma_m).$$

It can easily be shown that the left-hand side of (28) is regular at infinity. Hence it can be written as

$$\sum_{m=0}^{M} \mathbf{F}_m \frac{s}{s+\gamma_m} = \sum_{m=0}^{M} \mathbf{G}_m \frac{s}{s+\gamma_m} - \sum_{m=0}^{M} \mathbf{H}_m \frac{s}{s+\gamma_m}, \tag{29}$$

where the $\mathbf{F}_m$ are real (in general nonsymmetric) coefficient matrices,

$$0 = \gamma_0 < \gamma_1 < \gamma_2 \cdots < \gamma_M,$$

and the elements in $\mathbf{G}_m$ and $\mathbf{H}_m$ are nonnegative.

It is clear from (28) that the asymmetry in the $\mathbf{F}_m$ must be absorbed by the terms $-a\mathbf{Y}_{32} + b\mathbf{Y}_{32}{}'$. By equating the antisymmetric part of (29) to the antisymmetric part of (28), we obtain

$$\frac{b+a}{2} [\mathbf{Y}_{32}{}' - \mathbf{Y}_{32}] = \frac{1}{2} \sum_m \frac{s}{s+\gamma_m} [\mathbf{G}_m - \mathbf{G}_m{}' - \mathbf{H}_m + \mathbf{H}_m{}']. \tag{30}$$

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* The case in which a, $b < 0$ can be treated by an entirely analogous method, which involves interchanging the properties assigned to the matric polynomials $\mathbf{P}_1$ and $\mathbf{P}_2$ in (25). The required factorization can be obtained by factoring $\mathbf{P}'$ and taking the transpose of the resulting product.

Equation (30) is satisfied* with

$$\mathbf{Y}_{32} = -\frac{1}{a+b}\sum_m \frac{s}{s+\gamma_m}[\mathbf{G}_m + \mathbf{H}_m{}^t]. \tag{31}$$

The equation corresponding to (30) for the symmetric parts, with $\mathbf{Y}_{32}$ given by (31), is

$$\begin{aligned} ab\mathbf{Y}_{33} - \mathbf{Y}_{22} = \frac{b}{a+b}\sum_m \frac{s}{s+\gamma_m}[\mathbf{G}_m + \mathbf{G}_m{}^t] \\ - \frac{a}{a+b}\sum_m \frac{s}{s+\gamma_m}[\mathbf{H}_m + \mathbf{H}_m{}^t]. \end{aligned} \tag{32}$$

The identification of $\mathbf{Y}_{33}$ and $\mathbf{Y}_{22}$ can be made as follows:

$$\begin{aligned} \mathbf{Y}_{33} &\overset{od}{=} -\frac{1}{b(a+b)}\sum_m \frac{s}{s+\gamma_m}[\mathbf{H}_m + \mathbf{H}_m{}^t], \\ \mathbf{Y}_{22} &\overset{od}{=} -\frac{b}{(a+b)}\sum_m \frac{s}{s+\gamma_m}[\mathbf{G}_m + \mathbf{G}_m{}^t], \\ \mathbf{Y}_{33} &\overset{d}{=} \frac{1}{a(a+b)}\sum_m \frac{s}{s+\gamma_m}[\mathbf{G}_m + \mathbf{G}_m{}^t] + \sum_m \frac{s}{s+\gamma_m}\mathbf{J}_m, \\ \mathbf{Y}_{22} &\overset{d}{=} \frac{a}{a+b}\sum_m \frac{s}{s+\gamma_m}[\mathbf{H}_m + \mathbf{H}_m{}^t] + ab\sum_m \frac{s}{s+\gamma_m}\mathbf{J}_m, \end{aligned} \tag{33}$$

where "*od*" or "*d*" over an equal sign signifies that equality holds respectively only for the off-diagonal and on-diagonal elements. The diagonal matrices $\mathbf{J}_m$ in (33) are chosen to satisfy the dominant-diagonal condition for the last $2N$ rows of $\hat{\mathbf{Y}}$.

For the special case when all the $\mathbf{F}_m$ are symmetric matrices the structure can be simplified by setting $\mathbf{Y}_{32} = 0$. This leads to the identification:

$$\begin{aligned} \mathbf{Y}_{33} &\overset{od}{=} -\frac{1}{ab}\sum_m \mathbf{H}_m \frac{s}{s+\gamma_m}, \\ \mathbf{Y}_{22} &\overset{od}{=} -\sum_m \mathbf{G}_m \frac{s}{s+\gamma_m}, \\ \mathbf{Y}_{33} &\overset{d}{=} \frac{1}{ab}\sum_m \mathbf{G}_m \frac{s}{s+\gamma_m} + \sum_m \mathbf{J}_m \frac{s}{s+\gamma_m}, \\ \mathbf{Y}_{22} &\overset{d}{=} \sum_m \mathbf{H}_m \frac{s}{s+\gamma_m} + ab\sum_m \mathbf{J}_m \frac{s}{s+\gamma_m}. \end{aligned} \tag{34}$$

* There are, of course, other solutions of (30).

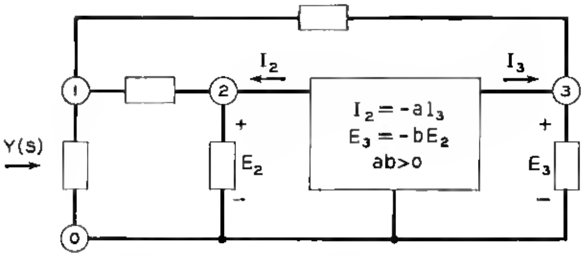

Fig. 3 — Realization of a general driving-point function.

Note that the elements in $\mathbf{Y}_{32}$ and tbe off-diagonal elements in $\mathbf{Y}_{22}$ and $\mathbf{Y}_{33}$ given by (33) and (34) are, as required, negative-RC driving-point admittance functions.

Hence, an arbitrary $N \times N$ matrix of real rational functions can be realized as the short-circuit admittance matrix of the structure shown in Fig. 2 in which the $(3N + 1)$-terminal network requires no internal nodes and contains only resistors and capacitors.* A numerical example is considered in Appendix B. The freedom implicit in the synthesis procedure can be exploited further to yield certain simplifications and other types of structures. Some of these possibilities may already have occurred to the sufficiently interested reader.

IV. DISCUSSION

In Section II it is shown that N is the sufficient and, in general, minimum number of negative-RC driving-point immittances that must be embedded in an N-port network of resistors and capacitors to realize as its immittance matrix an arbitrary symmetric $N \times N$ matrix of real rational functions in the complex-frequency variable.

Since any negative-RC driving-point admittance function which is regular at infinity can be written as the sum of a negative constant and an RL driving-point admittance function, it follows [recall from (16) that the y_i need not have a pole at infinity] that

Theorem:† *An arbitrary symmetric $N \times N$ matrix of real rational functions can be realized as the immittance matrix of an N-port transformerless RLC network containing N negative resistors. A canonical form is a $2N$-port network of resistors and capacitors terminated at each of N ports with an RL driving-point impedance in parallel with a negative resistor.*

* The complete structure for the special case $N = 1$ (and $\mathbf{Y}_{32} = 0$) is shown in Fig. 3.

† Carlin has established[21] some interesting related results for networks containing resistors, capacitors, inductors, gyrators, ideal transformers, and negative resistors.

The unbalanced realization of an N-port active RC network described in Section III leads to a particularly simple structural form for the required passive subnetwork. Possibilities of determining other structures are implicit in the method. An intriguing class of unsolved problems relate to the determination of structures which optimize some measure of performance such as the sensitivity function.

V. ACKNOWLEDGMENT

The writer is grateful to S. Darlington for his constructive criticism and advice.

APPENDIX A

Selection of $\mathbf{Y}_{11}$ *and Decomposition of* $\mathbf{P}$

The submatrix $\mathbf{Y}_{11}$ can be made to have dominant-diagonal coefficient matrices by choosing any realizable $N \times N$ RC admittance matrix, with elements of suitable degree as determined subsequently, and multiplying each diagonal entry by a sufficiently large positive real constant ρ. Denote the matrix determined in this way by

$$\mathbf{Y}_{11} = \frac{1}{q}\begin{bmatrix} \rho x_{11}' & x_{12} & \cdots & x_{1N} \\ \vdots & \rho x_{22}' & & \vdots \\ x_{N1} & & \cdots & \rho x_{NN}' \end{bmatrix}. \tag{35}$$

The polynomial det $\mathbf{P}$ can be written as

$$\det \mathbf{P} = \det\,[qN_{ij} - Dx_{ij}] = (-\rho)^N \left\{ D^N \prod_{i=1}^{N} x_{ii}' + \frac{R(s)}{\rho^N} \right\}, \tag{36}$$

where $R(s)/\rho^N$ is a polynomial with all coefficients that approach zero as ρ approaches infinity. We shall assume that deg x_{ii} = deg q ($i = 1,2,\cdots,N$), and that the x_{ii} are nonzero at the origin. Note that, as ρ approaches infinity, Ndeg q zeros of det $\mathbf{P}$ approach the zeros of

$$\prod_{i=1}^{N} x_{ii}'.$$

The zeros of this product can be chosen to be distinct and different from those of D. Hence, for a sufficiently large value of ρ, condition (c) of Section 2.1 is satisfied, and det $\mathbf{P}$ has at least N deg q distinct negative-real zeros that are different from those of q.

We next consider a sufficient condition for the removal of a linear factor of $\mathbf{P}$.

A.1 *Factorization of the Matric Polynomial* **P***

Let L be the degree of the highest degree polynomial in **P** and suppose that tbe zeros of

$$\det \mathbf{P} = \sum_{j=0}^{J} a_j s^j$$

include K distinct zeros.

Consider the result of determining a nonsingular matrix **Q** with constant elements such that every element in the ith column of **PQ** has a zero at $s = s_i$ $(i = 1,2,\cdots,N)$, where s_i is a zero of det **P**. If indeed this can be done, **P** can be written as

$$\mathbf{P} = (\mathbf{PQ})\mathbf{Q}^{-1} = \mathbf{P}'(\mathbf{D}\mathbf{Q}^{-1}), \tag{37}$$

where **D** is the diagonal matrix diag $[s - s_1, s - s_2, \cdots, s - s_N]$, and the degree of the highest degree polynomial in $\mathbf{P}'$ is $L - 1$. This is equivalent to removing a linear factor of the matric polynomial **P**:

$$\begin{aligned} \mathbf{P} = \sum_{j=1}^{L} s^j \mathbf{A}_j &= \left[\sum_{j=1}^{L-1} s^j \mathbf{A}_j'\right] \mathbf{D}\mathbf{Q}^{-1} \\ &= \left[\sum_{j=1}^{L-1} s^j \mathbf{A}_j' \mathbf{Q}^{-1}\right] \mathbf{Q}\mathbf{D}\mathbf{Q}^{-1} \\ &= \left[\sum_{j=1}^{L-1} s^j \mathbf{A}_j''\right] (s\mathbf{U} - \mathbf{B}), \end{aligned} \tag{38}$$

where **U** is the identity matrix of order N and

$$\mathbf{B} = \mathbf{Q} \operatorname{diag} [s_1,s_2,\cdots,s_N]\, \mathbf{Q}^{-1}.$$

We first develop a sufficient condition for the existence of a nonsingular matrix of constants $\mathbf{Q}_k$ such that every element in the kth column of $\mathbf{PQ}_k$ has a zero at $s = s_k$. It is then shown that **Q** can be constructed as the product of N matrices of this type.

At any zero of det **P**, say at $s = s_l$, the column rank of **P** is necessarily less than N, and hence there exists a relationship of the form

$$0 = \sum_{j=1}^{N} q_{jl} \mathbf{P}_j(s_l), \tag{39}$$

where $\mathbf{P}_j(s_l)$ is the jth column vector of **P** evaluated at $s = s_l$ and the

* The discussion is more general than is required for the purposes of this paper.

constants q_{jl} are not all zero. If, in addition, for some value k of the index l there exists a relationship of the form (39) with $q_{kk} \neq 0$, a matrix $\mathbf{Q}_k$ having the desired properties exists and in fact is given by

$$\mathbf{Q}_k = \begin{bmatrix} 1 & & & q_{1k} & & & \\ & \ddots & & q_{2k} & & & \\ & & 1 & \vdots & & & \\ & & & q_{kk} & & & \\ & & & \vdots & 1 & & \\ & & & & & \ddots & \\ & & & q_{Nk} & & & 1 \end{bmatrix}.$$

Consider

$$\det \mathbf{P} = \sum_{i=1}^{N} p_{ik}\Delta_{ik},$$

where the Δ_{ik} are the appropriate cofactors constructed from columns $1,2,\cdots,k-1, k+1,\cdots,N$ of det $\mathbf{P}$. Denote by $C_k(s)$ the polynomial which is the greatest common factor of all the Δ_{ik}. It follows that

$$\det \mathbf{P} = C_k(s) \sum_{i=1}^{N} p_{ik}\Delta_{ik}', \tag{40}$$

in which there are no factors common to all the Δ_{ik}'. It is evident that all $(N-1)$-rowed minors of det $\mathbf{P}$ constructed from columns $1,2,\cdots,k-1$, $k+1,\cdots,N$ cannot vanish at $s = s_k$, if s_k is a zero of

$$\sum_{i=1}^{N} p_{ik}\Delta_{ik}'$$

that is different from those of $C_k(s)$. In such cases the following set of equations yields only the trivial solution for the q_{jk}:

$$0 = \sum_{j \neq k}^{N} q_{jk}\mathbf{P}_j(s_k) \tag{41}$$

and hence

$$0 = \sum_{j=1}^{N} q_{jk}\mathbf{P}_j(s_k), \tag{42}$$

where $q_{kk} \neq 0$.

In other words, if det $\mathbf{P}$ has at least one zero which is different* from those of $C_k(s)$, a nonsingular matrix of constants, $\mathbf{Q}_k$, can be determined such that each element in the kth column of $\mathbf{PQ}_k$ has a zero at $s = s_k$.

Since the number of zeros of the polynomial $C_k(s)$ cannot exceed $(N - 1)L$, it is obviously sufficient that K, the number of distinct zeros of det $\mathbf{P}$, exceed $(N - 1)L$. Note that the degree of the highest degree polynomial in $\mathbf{P}$ and the zeros of det $\mathbf{P}$ are identical to the corresponding quantities in $\mathbf{PQ}_k$. Note also that the elements in all columns of $\mathbf{PQ}_k$ except the kth remain unchanged. Hence, if $K > (N - 1)L$, the matrix $\mathbf{Q}$ can be constructed as a product of N matrices $\mathbf{Q}_k$ chosen so that every element in the ith column of

$$\mathbf{P} \prod_{k=1}^{m} \mathbf{Q}_k, \qquad (i = 1,2,\cdots,m)$$

has a zero at $s = s_i$.

To summarize, if $(N - 1)L < K$, N zeros of Det. $\mathbf{P}$ can be removed as a linear factor of the matric polynomial $\mathbf{P}$. The remaining polynomial is of degree $L - 1$.†

The removal of a linear factor can be ensured under a weaker condition if $\mathbf{A}_L$, the leading coefficient of the matric polynomial, is singular. This matter is discussed in the following paragraph.

Let $\mathbf{R}$ be a nonsingular matrix of real constants chosen so that $\mathbf{A}_L\mathbf{R}$ has $N - r$ vanishing columns, where r is the rank of $\mathbf{A}_L$. Assume for the purposes of discussion that the last $N - r$ columns of $\mathbf{A}_L\mathbf{R}$ vanish. It follows that the elements in the last $N - r$ columns of $\mathbf{PR}$ have degrees not exceeding $L - 1$. In accordance with the discussion presented above, it is possible to determine a nonsingular matrix of constants $\mathbf{Q}_k$ such that each element in column k of $\mathbf{PRQ}_k$ has a zero at $s = s_k$ if det $\mathbf{P}$ has at least one zero that is different from those of $C_k'(s)$ [the greatest common factor of the $(N - 1)$-rowed minors of $\mathbf{PR}$ analogous to those of $\mathbf{P}$ above]. Note that if $1 \leqq k \leqq r$ the degree of $C_k'(s)$ cannot exceed

* A suitable $\mathbf{Q}_k$ corresponding to a multiple root of det $\mathbf{P}$ at $s = s_k$ can be determined if the nullity of $\mathbf{P}$ at $s = s_k$ exceeds the number of linearly independent nontrivial solutions for the q_{jk} in (41).

† This implies that the matric polynomial $\mathbf{P}$ can be written as

$$\mathbf{P} = \mathbf{C} \prod_{i=1}^{L} (s\mathbf{U} - \mathbf{B}_i),$$

when det $\mathbf{P}$ has NL distinct zeros. When these zeros are all real the coefficient matrices $\mathbf{C}$ and $\mathbf{B}_i$ are also real.

$(N - 1)L - (N - r)$. Therefore, if $K > (N - 1)L - (N = r)$, a nonsingular matrix

$$\mathbf{Q}' = \prod_{k=1}^{r} \mathbf{Q}_k$$

can certainly be determined such that each element in the kth column of $\mathbf{PRQ}'$ has a zero at $s = s_k$ $(k = 1,2,\cdots r)$, while each element in the last $N - r$ columns of $\mathbf{PRQ}'$ is of degree not exceeding $L - 1$. Hence, P can be written as follows:

$$\begin{aligned}\mathbf{P} &= (\mathbf{PRQ}')(\mathbf{RQ}')^{-1} \\ &= \mathbf{P}'' \operatorname{diag}[s - s_1, s - s_2, \cdots, s - s_r, \underbrace{1,1,\cdots,1}_{N-r}](\mathbf{RQ}')^{-1},\end{aligned} \tag{43}$$

$$\mathbf{P} = \mathbf{P}''[s\mathbf{F} + \mathbf{G}], \tag{44}$$

where $\mathbf{P}''$ is of degree $L - 1$ and $\mathbf{F}$ and $\mathbf{G}$ are constant $N \times N$ matrices. In particular, $\mathbf{F}$ is of rank r.

It should be clear that the factorization (44) is not dependent upon which $N - r$ columns of $\mathbf{A}_L\mathbf{R}$ vanish.

For our purposes it is sufficient to consider only the negative-real zeros of det $\mathbf{P}$. A moment's reflection will show that if $N\deg q$, the minimum number of distinct negative-real zeros of det $\mathbf{P}$, satisfies $N\deg q > (N - 1)L$, N distinct negative-real zeros of det $\mathbf{P}$ can be removed as a linear factor of $\mathbf{P}$. The remaining polynomial is of degree $L - 1$ and the matrix of constants $\mathbf{B}$ [in (38)] is real. It follows that Nk distinct negative-real zeros of det $\mathbf{P}$ can be removed as k linear factors if

$$(N - 1)[L - (k - 1)] < N\deg q - N(k - 1). \tag{45}$$

The degree of $\mathbf{P}$ is $L = \deg q + L_0$, where $L_0 = \max[\max \deg N_{ij}, \deg D]$. To ensure that $k = L_0$ linear factors of $\mathbf{P}$ can be removed, we have, from (45),

$$NL_0 - 1 < \deg q. \tag{46}$$

APPENDIX B

Synthesis of a Two-Port Network — A Numerical Example

To illustrate the main points in the synthesis technique presented in Section 3.1, we consider in detail the synthesis of a two-port network. Since the factorization of $\mathbf{P}$ is described elsewhere,[19] we select an example

for which it is possible to choose $\mathbf{Y}_{11}$ so that the required factoring is trivial. It is assumed that $a = b = 1$ [see (19) and (21)]:

Let the prescribed 2×2 matrix be

$$\mathbf{Y} = \frac{1}{D}[N_{ij}] = \frac{1}{s+3}\begin{bmatrix} 1 & s+3 \\ s-3 & 2 \end{bmatrix}. \tag{47}$$

The following matrix $\mathbf{Y}_{11}$ obviously satisfies the dominance condition with inequality:

$$\mathbf{Y}_{11} = \frac{1}{q}[x_{ij}] = \frac{1}{s+3}\begin{bmatrix} \rho(s+1) & 0 \\ 0 & \rho(s+2) \end{bmatrix}, \qquad \rho > 0. \tag{48}$$

Since $q = D$, the factorization of $\mathbf{P}$ is trivial. Specifically, we have

$$\mathbf{P} = (s+3)\begin{bmatrix} (1-\rho-\rho s) & s+3 \\ s-3 & (2-2\rho-\rho s) \end{bmatrix} = \mathbf{P}_1\mathbf{P}_2, \tag{49}$$

where

$$\mathbf{P}_1 = (s+3)\mathbf{U}, \qquad \mathbf{P}_2 = \begin{bmatrix} (1-\rho-\rho s) & s+3 \\ s-3 & (2-2\rho-\rho s) \end{bmatrix}, \tag{50}$$

and $\mathbf{U}$ is the identity matrix of order two. It is clear from (50) that ρ can be chosen so that det $\mathbf{P}_2$ has two distinct negative-real zeros. We choose $\rho = 10$, which yields

$$\mathbf{P}_2 = \begin{bmatrix} -(10s+9) & s+3 \\ s-3 & -(10s+18) \end{bmatrix},$$

$$\begin{aligned} \det \mathbf{P}_2 &= 99s^2 + 270s + 171 \\ &= 99(s+1.0000)(s+1.7273). \end{aligned} \tag{51}$$

Hence $\hat{\mathbf{Y}}$ will be of the form

$$\sum_{m=0}^{3} \mathbf{K}_m \frac{s}{s+\gamma_m}, \tag{52}$$

where $\gamma_0 = 0$, $\gamma_1 = 1.0000$, $\gamma_2 = 1.7273$, and $\gamma_3 = 3.0000$.

Since $\mathbf{P}_1$ is a diagonal matrix, $\mathbf{P}_3$ can be chosen to be a diagonal matrix. Let

$$\mathbf{P}_3 = -(s+2)\mathbf{U}. \tag{53}$$

Note that $(1/q)\mathbf{P}_3$ is a matrix of negative-RC driving-point admittances. Using (27), we can determine values of β_2/β_1 for which $\mathbf{Y}_{12}$ and $\mathbf{Y}_{13}$ are matrices of negative-RC driving-point admittances. Accordingly, with $\beta_2/\beta_1 = 0.5$ we obtain:

$$\begin{aligned}
\mathbf{Y}_{12} &= \frac{1}{\beta_2}\begin{bmatrix} -0.0833 & 0 \\ 0 & -0.0833 \end{bmatrix} \\
&\quad + \frac{s}{(s+3)\beta_2}\begin{bmatrix} -0.1666 & 0 \\ 0 & -0.1666 \end{bmatrix}, \\
\mathbf{Y}_{13} &= \frac{1}{\beta_2}\begin{bmatrix} -0.5833 & 0 \\ 0 & -0.5833 \end{bmatrix} \\
&\quad + \frac{s}{(s+3)\beta_2}\begin{bmatrix} -0.1666 & 0 \\ 0 & -0.1666 \end{bmatrix}.
\end{aligned} \tag{54}$$

From (48) with $\rho = 10$,

$$\mathbf{Y}_{11} = \begin{bmatrix} 3.3333 & 0 \\ 0 & 6.6666 \end{bmatrix} + \frac{s}{s+3}\begin{bmatrix} 6.6666 & 0 \\ 0 & 3.3333 \end{bmatrix}. \tag{55}$$

The choice $\beta_2 = 0.2$ satisfies the dominant-diagonal condition for the first two rows of $\mathbf{K}_0$ and $\mathbf{K}_3$. This condition is satisfied with the equality sign in the first row of $\mathbf{K}_0$, and for this reason reduces by one the number of resistors necessary to realize $\mathbf{K}_0$.

Using $(99\beta_1\beta_2)^{-1} = 0.1263$, we obtain from (28), (29), (51), and (53),

$$\begin{aligned}
&\frac{0.1263(s+2)}{(s+1.0000)(s+1.7273)}\begin{bmatrix} 10s+18 & s+3 \\ s-3 & 10s+9 \end{bmatrix} \\
&\qquad = \sum_{m=0}^{2} \mathbf{F}_m \frac{s}{s+\gamma_m}.
\end{aligned} \tag{56}$$

Equation (56) can be expressed as

$$\begin{aligned}
\sum_{m=0}^{2} \mathbf{F}_m \frac{s}{s+\gamma_m} &= \begin{bmatrix} 2.6316 & 0.4386 \\ -0.4386 & 1.3159 \end{bmatrix} \\
&\quad + \frac{s}{s+1.0000}\begin{bmatrix} -1.3889 & -0.3472 \\ 0.6944 & 0.17361 \end{bmatrix} \\
&\quad + \frac{s}{s+1.7273}\begin{bmatrix} 0.0199 & 0.0348 \\ -0.1296 & -0.2267 \end{bmatrix}.
\end{aligned} \tag{57}$$

The coefficient matrices $\mathbf{K}_m$ can readily be constructed with the aid of (31), (33), (54), (55), and (57). Consider for example $\mathbf{K}_0$. From (57),

$$\mathbf{G}_0 = \begin{bmatrix} 2.6315 & 0.4386 \\ 0 & 1.3159 \end{bmatrix}, \qquad \mathbf{H}_0 = \begin{bmatrix} 0 & 0 \\ 0.4386 & 0 \end{bmatrix}. \tag{58}$$

Using (31), (33), and (58),

$$\mathbf{Y}_{32_0} = \begin{bmatrix} -1.3157 & -0.4386 \\ 0 & -0.6579 \end{bmatrix},$$

$$\mathbf{Y}_{33_0} = \begin{bmatrix} 2.6315 + j_{10} & -0.2193 \\ -0.2193 & 1.3159 + j_{20} \end{bmatrix}, \tag{59}$$

$$\mathbf{Y}_{22_0} = \begin{bmatrix} j_{10} & -0.2193 \\ -0.2193 & j_{20} \end{bmatrix},$$

where j_{10} and j_{20} are the diagonal elements in $\mathbf{J}_0$ [see (33)].

From (54) with $\beta_2 = 0.2$, (55) and (59)

$$K_0 = \left|\begin{array}{cc|cc|cc} 3.3333 & 0 & -0.4166 & 0 & -2.9166 & 0 \\ 0 & 6.6666 & 0 & -0.4166 & 0 & -2.9166 \\ \hline -0.4166 & 0 & j_{10} & -0.2193 & -1.3157 & 0 \\ 0 & -0.4166 & -0.2193 & j_{20} & -0.4386 & -0.6579 \\ \hline -2.9166 & 0 & -1.3157 & -0.4386 & 2.6315 + j_{10} & -0.2139 \\ 0 & -2.9166 & 0 & -0.6579 & -0.2139 & 1.3159 + j_{20} \end{array}\right|. \tag{60}$$

It is easy to verify that the choice $j_{01} = 2.2534$, $j_{02} = 2.4725$ satisfies the dominant-diagonal condition for the last four rows of $\mathbf{K}_0$ and in particular satisfies the condition with equality in rows five and six.

The remaining coefficient matrices $\mathbf{K}_1$, $\mathbf{K}_2$, and $\mathbf{K}_3$ can be constructed in a similar manner. The realization of the matrix $\hat{\mathbf{Y}}$ is straightforward.

B.1 *An Alternative Synthesis*

Some reflection will show that a large number of elements are required to realize $\hat{\mathbf{Y}}$. This number can be reduced by choosing the elements in $\mathbf{P}_3$ differently. For this reason it is worth while to consider the following alternative synthesis technique.

If $\mathbf{P}$ could be written as $\mathbf{P}_1'\mathbf{P}_2'$, where $\mathbf{P}_2'$ has the properties previously associated with $\mathbf{P}_3$ (here denoted by $\mathbf{P}_3'$), the sum in (29), with $\mathbf{P}_2' = \mathbf{P}_3'$, would contain simply one term [see (28) and recall that $D = q$] while the properties assigned to $\mathbf{Y}_{12}$ and $\mathbf{Y}_{13}$ are permitted to remain invariant.

Consider the matrix $\mathbf{P}_2$ given in (51) and repeated below for convenience:

$$\mathbf{P}_2 = \begin{bmatrix} -(10s + 9) & s + 3 \\ s - 3 & -(10s + 18) \end{bmatrix}. \tag{61}$$

By adding the first row of $\mathbf{P}_2$ to the second, and then adding the new second row to the first, we obtain

$$\mathbf{P}_2' = \begin{bmatrix} -(19s + 21) & -(8s + 12) \\ -(9s + 12) & -(9s + 15) \end{bmatrix}. \tag{62}$$

Note that each element in $(1/q)\mathbf{P}_2'$ is a negative-RC driving-point admittance function. Since $\mathbf{P}_2'$ can be obtained from $\mathbf{P}_2$ by successive elementary operations on rows, the relation between $\mathbf{P}_2$ and $\mathbf{P}_2'$ can be expressed by

$$\mathbf{P}_2' = \mathbf{T}\mathbf{P}_2, \tag{63}$$

where $\mathbf{T}$ is a 2×2 nonsingular matrix of real constants. Specifically,

$$\mathbf{T} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}. \tag{64}$$

The matrix $\mathbf{P}$ can be written as

$$\mathbf{P} = \mathbf{P}_1\mathbf{T}^{-1}\mathbf{T}\mathbf{P}_2 = \mathbf{P}_1'\mathbf{P}_2', \tag{65}$$

where $\mathbf{P}_2' = \mathbf{T}\mathbf{P}_2$ and $\mathbf{P}_1' = \mathbf{P}_1\mathbf{T}^{-1}$. Using (50), (64), and (65),

$$\mathbf{P}_1' = \begin{bmatrix} (s+3) & -(s+3) \\ -(s+3) & 2(s+3) \end{bmatrix}, \tag{66}$$

At this point we let $\mathbf{P}_3' = \mathbf{P}_2'$ and return to the procedure demonstrated earlier.

From (27) with $\beta_2/\beta_1 = 1$, (62), (66), and (55),

$$\begin{aligned} \mathbf{Y}_{12} &= \frac{1}{\beta_2}\begin{bmatrix} -3.0 & -2.5 \\ -2.5 & -1.5 \end{bmatrix} + \frac{s}{\beta_2(s+3)}\begin{bmatrix} -6.0 & -2.5 \\ -2.0 & -2.0 \end{bmatrix}, \\ \mathbf{Y}_{13} &= \frac{1}{\beta_2}\begin{bmatrix} -4.0 & -1.5 \\ -1.5 & -3.5 \end{bmatrix} + \frac{s}{\beta_2(s+3)}\begin{bmatrix} -6.0 & -2.5 \\ -2.0 & -2.0 \end{bmatrix}, \\ \mathbf{Y}_{11} &= \begin{bmatrix} 3.3333 & 0 \\ 0 & 6.6666 \end{bmatrix} + \frac{s}{s+3}\begin{bmatrix} 6.6666 & 0 \\ 0 & 3.3333 \end{bmatrix}. \end{aligned} \tag{67}$$

The dominance condition is satisfied in the first and second rows of $\mathbf{K}_0'$ and $\mathbf{K}_1'$ with $\beta_2 = 3.3000$. The condition is satisfied with equality in the first row of $\mathbf{K}_0'$.

The left-hand side of (28) is

$$\frac{1}{\beta_1\beta_2}\frac{D}{q}\mathbf{P}_3'\mathbf{P}_2'^{-1} = 0.0918\,\mathbf{U}, \tag{68}$$

where $\mathbf{U}$ is the identity matrix of order two.

Equations (34) and (68) lead to

$$\begin{aligned} \mathbf{Y}_{33} &= \begin{bmatrix} 0.0918 + j_{10}' & 0 \\ 0 & 0.0918 + j_{20}' \end{bmatrix} + \frac{s}{s+3}\begin{bmatrix} j_{11}' & 0 \\ 0 & j_{21}' \end{bmatrix}, \\ \mathbf{Y}_{22} &= \begin{bmatrix} j_{10}' & 0 \\ 0 & j_{20}' \end{bmatrix} + \frac{s}{s+3}\begin{bmatrix} j_{11}' & 0 \\ 0 & j_{21}' \end{bmatrix}, \\ \mathbf{Y}_{32} &= 0. \end{aligned} \tag{69}$$

The coefficient matrix $\mathbf{K}_0'$ is

$$
\mathbf{K}_0' = \left|\begin{array}{cc|cc|cc}
3.3333 & 0 & -0.9091 & -0.7576 & -1.2121 & -0.4545 \\
0 & 6.6666 & -0.7576 & -0.4545 & -0.4545 & -1.0606 \\
\hline
-0.9091 & -0.7576 & j_{10}' & 0 & 0 & 0 \\
-0.7576 & -0.4545 & 0 & j_{20}' & 0 & 0 \\
\hline
-1.2121 & -0.4545 & 0 & 0 & 0.0918 + j_{10}' & 0 \\
-0.4545 & -1.0606 & 0 & 0 & 0 & 0.0918 + j_{20}'
\end{array}\right|
$$

The dominance condition is satisfied in the last four rows of $\mathbf{K}_0'$ (satisfied with equality in the third and sixth rows) with $j_{10}' - 1.6667$, $j_{20}' = 1.4233$.

The remaining coefficient matrix $\mathbf{K}_1'$ is given by

$$
\mathbf{K}_1' = \left|\begin{array}{cc|cc|cc}
6.6666 & 0 & -1.8182 & -0.7576 & -1.8182 & -0.7576 \\
0 & 3.3333 & -0.6061 & -0.6061 & -0.6061 & -0.6061 \\
\hline
-1.8182 & -0.6061 & j_{11}' & 0 & 0 & 0 \\
-0.7576 & -0.6061 & 0 & j_{21}' & 0 & 0 \\
\hline
-1.8182 & -0.6061 & 0 & 0 & j_{11}' & 0 \\
-0.7576 & -0.6061 & 0 & 0 & 0 & j_{21}'
\end{array}\right|
$$

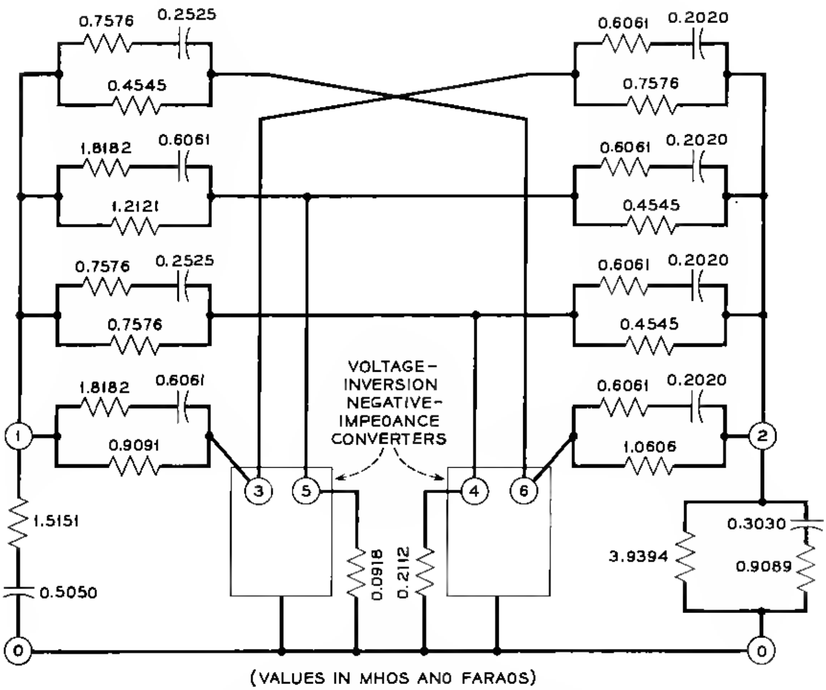

Fig. 4 — Realization of two-port network example.

For this matrix the dominance condition is satisfied with $j_{11}' = 2.4243$, $j_{21}' = 1.3637$.

The final network is shown in Fig. 4.

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